انتگرال سکانت (و توان‌های آن)

در این نوشتار قصد داریم تا انتگرال توابع $\sec(x)$، $\sec^2(x)$، ...، $\sec^n(x)$ را آموزش دهیم. از آن‌جا که نحوه‌ی محاسبه‌ی انتگرال توان‌های زوج سکانت با توان‌های فرد آن تفاوت دارد، بنابراین ابتدا به‌برآورد انتگرال توان‌های زوج سکانت (که ساده‌تر از توان‌های فرد است)، می‌پردازیم.

- $\int \sec^2(x)\,dx = \int (1+\tan^2(x))\,dx = \tan(x) + C.$

- $\int \sec^4(x)\,dx = \int \sec^2(x)\sec^2(x)\,dx = \int \sec^2(x)\,(1+\tan^2(x))dx = \int (\sec^2(x) + \sec^2(x)\tan^2(x))dx = \int \sec^2(x)\,dx + \int \sec^2(x)\tan^2(x)\,dx = \tan(x) + \int (\tan(x))'\tan^2(x)\,dx = \tan(x) + \frac{1}{3}\tan^3(x) + C.$

- $\int \sec^6(x)\,dx = \int \sec^2(x)\sec^4(x)\,dx = \int \sec^2(x)\,(1+\tan^2(x))^2 dx = \int \sec^2(x)\,(1 + 2\tan^2(x) + \tan^4(x))\,dx = \int (\sec^2(x) + 2\sec^2(x)\tan^2(x) + \sec^2(x)\tan^4(x))dx = \int \sec^2(x)\,dx + 2\int \sec^2(x)\tan^2(x)\,dx + \int \sec^2(x)\tan^4(x)\,dx = \tan(x) + 2\int ((\tan(x))'\tan^2(x))\,dx + \int (\tan(x))'\tan^4(x)\,dx = \tan(x) + \frac{2}{3}\tan^3(x) + \frac{1}{5}\tan^5(x) + C.$

- $\int \sec^8(x)\,dx = \int \sec^2(x)\sec^6(x)\,dx = \int \sec^2(x)\,(1+\tan^2(x))^3 dx = \int \sec^2(x)\,(1 + 3\tan^2(x) + 3\tan^4(x) + \tan^6(x))\,dx = \int (\sec^2(x) + 3\sec^2(x)\tan^2(x) + 3\sec^2(x)\tan^4(x) + \sec^2(x)\tan^6(x))dx = \int \sec^2(x)\,dx + 3\int \sec^2(x)\tan^2(x)\,dx + 3\int \sec^2(x)\tan^4(x)\,dx + \int \sec^2(x)\tan^6(x)\,dx = \tan(x) + 3\int ((\tan(x))'\tan^2(x))\,dx + 3\int (\tan(x))'\tan^4(x)\,dx + \int (\tan(x))'\tan^6(x)\,dx = \tan(x) + \tan^3(x) + \frac{3}{5}\tan^5(x) + \frac{1}{5}\tan^7(x) + C.$

- $\int \sec^{2n}(x)\,dx = \tan(x) + \frac{\binom{n-1}{1}}{3}\tan^3(x) + \frac{\binom{n-1}{2}}{5}\tan^5(x) + \frac{\binom{n-1}{3}}{7}\tan^7(x) + \cdots + \frac{\binom{n-1}{n-1}}{2n-1}\tan^{2n-1}(x) + C.$

انتگرال توان‌های فرد سکانت را به‌صورت بازگشتی با بهره‌گیری از روش انتگرال پاره‌ای یا جزبه‌جز می‌یابیم.

- $\int \sec(x)\,dx = \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}dx = \int \frac{\sec(x)\tan(x)+\sec^2(x)}{\sec(x)+\tan(x)}dx = \int \frac{(\sec(x)+\tan(x))'}{\sec(x)+\tan(x)}dx = \ln|(\sec(x)+\tan(x))| + C.$

- $\int \sec^3(x)\,dx = \int \overbrace{\sec(x)}^{u}\overbrace{\sec^2(x)}^{dv}dx = \overbrace{\sec(x)}^{u}\overbrace{\tan(x)}^{v} - \int \overbrace{\tan(x)}^{v}\overbrace{\sec(x)\tan(x)}^{du}dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx = \sec(x)\tan(x) - \int \sec(x)(\sec^2(x)-1)\,dx = \sec(x)\tan(x) - \int(\sec^3(x)-\sec(x))\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx + \ln|\sec(x)+\tan(x)| \Rightarrow 2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| \Rightarrow \int \sec^3(x)\,dx = \frac{1}{2}\{\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\} + C.$

- $\int \sec^5(x)\,dx = \int \overbrace{\sec(x)}^{u}\overbrace{\sec^4(x)}^{dv}dx = \overbrace{\sec(x)}^{u}\left(\overbrace{\tan(x)+\frac{1}{3}\tan^3(x)}^{v}\right) - \int\left(\overbrace{\tan(x)+\frac{1}{3}\tan^3(x)}^{v}\right)\overbrace{\sec(x)\tan(x)}^{du}dx = \sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) - \int \sec(x)\tan^2(x)\,dx - \frac{1}{3}\int \sec(x)\tan^4(x)\,dx = \sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) - \int \sec(x)(\sec^2(x)-1)\,dx - \frac{1}{3}\int \sec(x)\overbrace{(\sec^2(x)-1)^2}^{(\tan^2(x))^2}dx = \sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) - \int(\sec^3(x)-\sec(x))\,dx - \frac{1}{3}\int \sec(x)(\sec^4(x)-2\sec^2(x)+1)dx = \sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx - \frac{1}{3}\int \sec^5(x)\,dx + \frac{2}{3}\int \sec^3(x)\,dx - \frac{1}{3}\int \sec(x)\,dx = \sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) - \frac{1}{2}\{\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\} + \ln|\sec(x)+\tan(x)| - \frac{1}{3}\int \sec^5(x)\,dx + \frac{2}{3}\times\frac{1}{2}\{\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\} - \frac{1}{3}\ln|\sec(x)+\tan(x)| \Rightarrow \left(1+\frac{1}{3}\right)\int \sec^5(x)\,dx = \left(1-\frac{1}{2}+\frac{1}{3}\right)\sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) + \left(-\frac{1}{2}+1+\frac{1}{3}-\frac{1}{3}\right)\ln|\sec(x)+\tan(x)| \Rightarrow \frac{4}{3}\int \sec^5(x)\,dx = \frac{5}{6}\sec(x)\tan(x) + \frac{1}{3}\sec(x)\tan^3(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| \Rightarrow \int \sec^5(x)\,dx = \frac{5}{8}\sec(x)\tan(x) + \frac{1}{4}\sec(x)\tan^3(x) + \frac{3}{8}\ln|\sec(x)+\tan(x)| + C.$

- $\int \sec^7(x)\,dx = \int \overbrace{\sec(x)}^{u}\overbrace{\sec^6(x)}^{dv}\,dx = \overbrace{\sec(x)}^{u}\left(\overbrace{\tan(x)+\frac{2}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)}^{v}\right) - \int\left(\overbrace{\tan(x)+\frac{2}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)}^{v}\right)\overbrace{\sec(x)\tan(x)}^{du}\,dx = \sec(x)\tan(x) + \frac{2}{3}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) - \int \sec(x)\tan^2(x)\,dx - \frac{2}{3}\int \sec(x)\tan^4(x)\,dx - \frac{1}{5}\int \sec(x)\tan^6(x)\,dx = \sec(x)\tan(x) + \frac{2}{3}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) - \int \sec(x)\left(\sec^2(x)-1\right)dx - \frac{2}{3}\int \sec(x)\overbrace{\left(\sec^2(x)-1\right)^2}^{\left(\tan^2(x)\right)^2}dx - \frac{1}{5}\int \sec(x)\overbrace{\left(\sec^2(x)-1\right)^3}^{\left(\tan^2(x)\right)^3}dx = \sec(x)\tan(x) + \frac{2}{3}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) - \int\left(\sec^3(x)-\sec(x)\right)dx - \frac{2}{3}\int \sec(x)\left(\sec^4(x)-2\sec^2(x)+1\right)dx - \frac{1}{5}\int \sec(x)\left(\sec^6(x)-3\sec^4(x)+3\sec^2(x)-1\right)dx = \sec(x)\tan(x) + \frac{2}{3}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx - \frac{2}{3}\int \sec^5(x)\,dx + \frac{4}{3}\int \sec^3(x)\,dx - \frac{2}{3}\int \sec(x)\,dx - \frac{1}{5}\int \sec^7(x)\,dx + \frac{3}{5}\int \sec^5(x)\,dx - \frac{3}{5}\int \sec^3(x)\,dx + \frac{1}{5}\int \sec(x)\,dx = \sec(x)\tan(x) + \frac{2}{3}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) - \frac{4}{15}\int \sec^3(x)\,dx + \frac{8}{15}\int \sec(x)\,dx - \frac{1}{15}\int \sec^5(x)\,dx - \frac{1}{5}\int \sec^7(x)\,dx = \sec(x)\tan(x) + \frac{2}{3}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) - \frac{4}{15}\left\{\frac{1}{2}\{\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\}\right\} + \frac{8}{15}\ln|\sec(x)+\tan(x)| - \frac{1}{15}\left\{\frac{5}{8}\sec(x)\tan(x) + \frac{1}{4}\sec(x)\tan^3(x) + \frac{3}{8}\ln|\sec(x)+\tan(x)|\right\} - \frac{1}{5}\int \sec^7(x)\,dx = \left(1-\frac{2}{15}-\frac{1}{24}\right)\sec(x)\tan(x) + \left(\frac{2}{3}-\frac{1}{60}\right)\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) + \left(-\frac{4}{15}+\frac{8}{15}-\frac{1}{40}\right)\ln|\sec(x)+\tan(x)| - \frac{1}{5}\int \sec^7(x)\,dx = \frac{33}{40}\sec(x)\tan(x) + \frac{13}{20}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) + \frac{29}{120}\ln|\sec(x)+\tan(x)| - \frac{1}{5}\int \sec^7(x)\,dx \Rightarrow \frac{6}{5}\int \sec^7(x)\,dx = \frac{33}{40}\sec(x)\tan(x) + \frac{13}{20}\sec(x)\tan^3(x) + \frac{1}{5}\sec(x)\tan^5(x) + \frac{29}{120}\ln|\sec(x)+\tan(x)| \Rightarrow \int \sec^7(x)\,dx = \frac{11}{16}\sec(x)\tan(x) + \frac{13}{24}\sec(x)\tan^3(x) + \frac{1}{6}\sec(x)\tan^5(x) + \frac{29}{144}\ln|\sec(x)+\tan(x)| + C$

- $\int \sec^{2n+1}(x)\,dx = \frac{b}{a}\sec(x)\tan(x) + \frac{d}{c}\sec(x)\tan^3(x) + \cdots + \frac{e}{f}\sec(x)\tan^{2n-1}(x) + \frac{h}{g}\ln|\sec(x)+\tan(x)| + C$

- $a_n = \int \sec^{2n+1}(x)\,dx \quad , \quad b_n = \tan(x)\sec^{2n+1}(x) \quad , \quad a_1 = \int \sec^3(x)\,dx \Rightarrow$
  $a_n = \sum_{i=1}^{n-1} B_{n-i} b_{n-i} + A_1 a_1 \quad , \quad B_{n-i} = \frac{(2n-1)!((n-i)!)^2}{2^{2i-1} n!(n-1)!(2n-2i+1)!} \quad , \quad A_1 = \frac{(2n-1)!}{2^{2n-2} n!(n-1)!} \Rightarrow$
  $\int \sec^{2n+1}(x)\,dx = \frac{(2n-1)!}{n!(n-1s)!} \tan(x) \sum_{i=1}^{n-1} \frac{((n-i)!)^2}{2^{2i-1}(2n-2i+1)!} \sec^{2n-2i+1}(x) +$
  $\frac{(2n-1)!}{2^{2n-1} n!(n-1)!} (\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + C$